Lec23 (Radon-Nikodym Theorem)

Let $f \geq 0$ and define on $(X, \mathcal{F}, \mu)$

$$\nu(E) = \int_E f \, d\mu \qquad \forall \; E \in \mathcal{F}.$$

Then $\nu(E)$ is a meas. It satisfies the following prop:

$$\mu(E) = 0 \Rightarrow \int_E f \, d\mu = 0 \qquad - \; ⋆$$

(Suppose $f = +\infty$. Then could be a problem. Pick $\varphi \leq f$ simple supported on E then $\int_E \varphi = 0 \quad \forall \varphi$

$\Rightarrow \int_E f = 0$)

We use ⋆ as the definition of absolute continuity.

Prop 19: let $(X, \mathcal{F}, \mu)$ be a measure, and ν finite. Then ν is AC wrt μ iff $\forall \; \epsilon > 0 \; \exists \; \delta$ st

$$\mu(E) < \delta \Rightarrow \nu(E) < \epsilon \qquad 2⋆$$

Pf: It's clear that if 2⋆ holds and $\mu(E) = 0$ then $\nu(E) < \epsilon \quad \forall \; \epsilon$ and hence $\nu(E) = 0$

For the converse, suppose $\exists\ E_n$ st $\mu(E_n) < \frac{1}{2^n}$

and $\nu(E_n) \geq \epsilon_0 \quad \forall n.$

Then $\mu\left(\bigcup_{n=k}^{\infty} E_n\right) \leq \frac{1}{2^{k-1}} \quad \forall k \geq 1$

and $B_k = \bigcup_{n=k}^{\infty} E_n$ is decreasing in k. Thus

$\bigcap B_k$ is such that

$\mu\left(\bigcap_k B_k\right) = 0$ and $\nu\left(\bigcap_k B_k\right) \geq \epsilon_0$

To show $\nu\left(\bigcap_k B_k\right) \geq \epsilon_0$ we need that

$B_1 = \bigcup_{k=1}^{\infty} E_k$ has finite ν measure if we want to use

continuity. But ν is finite, thus $\nu\left(\bigcap_k B_k\right) \geq \epsilon_0$

EX: $\mu = \text{Leb}[0,1] \quad \frac{\partial \nu}{d\mu} = \frac{1}{x}$ on $(0,1]$

Then if $\mu(E) = 0$ Then $\int \frac{1}{x} 1_E = 0$ Take any simple fn on a meas 0 set, it will always integrate to 0

$\Rightarrow \nu \ll \mu$.

★ Good exam problem

Now to show $\exists\ \epsilon > 0$ st $\forall \delta > 0\ \exists\ E\ \ \mu(E) < \delta$ but $\nu(E) \geq \epsilon_0$

Choose $\epsilon = 1$, and for any $\delta > 0$ choose $E = [0, \delta/2]$

$\nu(E) = +\infty$

Radon-Nikodym derivative

If h is AC on $[a,b]$ then

$$h(d) - h(c) = \int_c^d h' \qquad \text{by FTC.}$$

Associated with any h AC on $[a,b]$ we can define a signed measure on Borel measurable sets E

$$\nu(E) = \int_E h'$$

It's obvious that $\nu << m$ (Lebesgue measure on Borel sets)

Conversely, suppose we're given ν signed and finite st $\nu << m$. Define

$$h(x) = \nu([a,x)) = \nu_1([a,x)) - \nu_2([a,x))$$

h is AC by previous and therefore

$$h(d) - h(c) = \nu([c,d]) = \int_c^d h' \, dm$$

In fact we can show directly that for any Borel E

$$\nu(E) = \int_E h' \, dm$$

Thus h' is the Radon-Nikodym derivative of ν wrt Lebesgue measure. We seek to generalize this construction.

Radon Nikodym Theorem

Suppose ν is any absolutely continuous σ-finite measure wrt another σ-finite measure μ.

Then $\exists\ f \geq 0$ st $\nu(E) = \int f \, d\mu$

Moreover f is unique a.e μ.

Pf: σ finiteness implies $\exists\ \bigcup_n X_n = X$ $\mu(X_n) < \infty$ and $\nu(X_n) < \infty$. There is a bit of a subtlety to show that $\exists\ X_n \uparrow$ $\mu(X_n) < \infty$ $\nu(X_n) < \infty$ and $\bigcup X_n = X$. $\exists\ A_n, B_n \uparrow X$ st $\mu(A_n), \nu(B_n) < \infty$. So choose $X_n = A_n \cap B_n$. If $x \in X$ then $x \in A_n$ and $x \in B_m \Rightarrow x \in X_{max(m,n)}$. $\Rightarrow \bigcup X_n = X$

So we may assume $\mu(X), \nu(X) < \infty$

Assume $\nu \not\equiv 0$ (not identically 0)

Will show that

Good in class exercise.

$\exists\ f$ st $\int_X f\,d\mu > 0$ and $\nu(E) \leq \int_E f\,d\mu$

Consider for $\lambda > 0$ $\nu - \lambda\mu$. This is a signed meas

(since ν is finite). By Hahn-Jordan $\exists\ P_\lambda, N_\lambda$ st

$P_\lambda \cap N_\lambda = \phi$ and P_λ positive, N_λ negative for $\nu - \lambda\mu$.

<u>Claim</u>: $\exists\ \lambda > 0$ st $\mu(P_\lambda) > 0$

Pf: Suppose not, then $\mu(P_\lambda) = 0$

if $E \subset P_\lambda$, $\mu(E) \leq \mu(P_\lambda) = 0$

Therefore for any $E \in \mathcal{F}$,

$$\nu(E) - \lambda\mu(E) = \nu(E \cap P_\lambda) + \nu(E \cap N_\lambda) - \lambda\left[\mu(E \cap P_\lambda) + \mu(E \cap N_\lambda)\right]$$

$= \nu(E \cap N_\lambda) - \lambda\mu(E \cap N_\lambda) \leq 0$ since

N_λ is negative for $\nu - \lambda\mu$

But this means

$$\nu(E) \leq \lambda\mu(E) \quad \forall\ E \in \mathcal{F} \text{ and } \lambda > 0$$

So $\nu(E) = 0$ $\forall$ $\mu(E) > 0$ and $\nu(E) = 0$

if $\mu(E) = 0$ by AC! Thus $\nu \equiv 0$

So assume that $\exists\ \lambda$ st $\mu(P_\lambda) > 0$

and let $f = \lambda \mathbb{1}_{P_\lambda}$

Then $\int_E f \, d\mu = \lambda \mu(E \cap P_\lambda) \leq \nu(E \cap P_\lambda)$

$\leq \nu(E)$

since $E \cap P_\lambda$ is positive

So there is at least one function that is such that

$$\int_E f \, d\mu \leq \nu(E) \quad \forall E \in \mathcal{F}$$

So define

$$M = \sup \Big[\int_X f \, d\mu \;\Big|\; \underbrace{\int_E f \leq \nu(E) \; \forall E}_{\mathcal{G}} \Big]$$

$M < \infty$ since ν is finite.

Choose $\{f_n\}$ st M is achieved by it, and note that

$$\int_E \max(g, f) = \int_{E \cap \{g > f\}} g + \int_{\{g \leq f\} \cap E} f$$

$$\leq \nu(E \cap \{g > f\}) + \nu(\{g \leq f\} \cap E)$$

$$\leq \nu(E)$$

$\Rightarrow \mathcal{G}$ is closed under max operation. So replace

$\{f_n\}$ by $\hat{f}_n = \max\{f_1, \dots, f_n\}$ and gen

an increasing seq st

$$\int_X \hat{f}_n = M$$

Since $\hat{f}_n \uparrow$ we can define $f = \lim_{n \to \infty} \hat{f}_n$ meas.

This convergence is EVERYWHERE, so f is meas.

By MCT $\lim_{n\to\infty} \int_E \hat{f}_n \leq \int_E f \leq \nu(E)$

so f also is in $\mathcal{G}$.

To show that $\nu(E) = \int_E f \, d\mu \quad \forall E$.

Suppose not and define

$$\eta(E) = \nu(E) - \int_E f \, d\mu.$$

$\eta(E)$ is a signed meas. and since $f \in \mathcal{G}$ $\eta(E) \geq 0$ $\forall E \in \mathcal{F}$. Further η is AC wrt μ so if $\eta \neq 0$ then $\exists$ $\hat{f}$ st $\int_X \hat{f} > 0$ and $\int_E \hat{f} \leq \eta(E)$

$\hookrightarrow$ recall we chose $f = 1_{P_\lambda} \lambda$ and $\mu(P_\lambda) > 0$ $\Rightarrow \int f > 0$.

$= \nu(E) - \int_E f \quad \forall E \Rightarrow f + \hat{f} \in \mathcal{G}$ and $\int_X f + \hat{f} > M$ which is a contradiction.

Uniqueness: $\nu(E) = \int_E f_1 \, d\mu = \int_E f_2 \, d\mu \quad \forall E \in \mathcal{F}$

By previous $f_1 = f_2$ a.e. μ.

☆ worth giving on final.

Remark: σ-finiteness is necessary. There is an example where you have $\nu << \mu$, but μ is not σ-finite and hence there is no f such that $\nu = \int f$

Suppose ν is signed. Then $\exists$ Jordan decomposition $\nu = \nu_1 - \nu_2$. Then as before $|\nu| = \nu_1 + \nu_2$. We say ν is AC wrt μ if $|\nu|$ is AC wrt μ. $\Rightarrow \nu_1$ and ν_2 are also AC wrt μ. So $\exists f, g$ $\quad \nu_1 = \int f \quad \nu_2 = \int g$

Moreover since ν is signed one of ν_1, ν_2 is finite. So $\nu = \int f - g$ is well-defined.

Mutually Singular measures Given $(X, \mathcal{F})$; two meas. μ and ν are singular if

$$X = A \sqcup B \qquad \mu(B) = 0 \quad \text{and } \nu(A) = 0$$

Write $\quad \mu \perp \nu$

Lebesgue decomposition Theorem: Suppose $(X, \mathcal{F}, \mu)$ σ-finite and ν is also σ-finite. Then $\exists$ ν_1 and ν_2 st

$$\nu = \nu_1 + \nu_2 \quad , \quad \nu_1 << \mu \quad \text{and} \quad \nu_2 \perp \mu$$

The decomposition is unique.

Pf: Let $\lambda = \mu + \nu$, then

$\int g\,d\lambda = \int g\,d\mu + \int g\,d\nu$ ☆ HW problem

Then $\mu << \lambda$ so $\exists f$ st

$$\mu(E) = \int f\,d\lambda = \underbrace{\int_E f\,d\mu}_{} + \int_E f\,d\nu$$

This gives an indication of the AC part.

Lets look at $X_+ = \{x \mid f > 0\}$

$X_0 = \{x \mid f = 0\}$

So $\nu_+(E) = \nu(E \cap X_+)$ and $\nu_0(E) = (E \cap X_0)$

ν_+ and ν_0 are obviously mutually singular.

$\nu_+(X_0) = 0 \qquad \nu_0(X_+) = 0$

How to show $\nu <<$ and $\nu \perp \mu$.

Obviously $\mu(X_0) = \int_{X_0} f\,d\mu + \int_{X_0} f\,d\nu = 0$

since $f = 0$ on X_0

and $\nu_2(X_+) = 0$ so $\nu_2 \perp \mu$

If $\mu(E) = 0$ then $\int_E f\,d\mu = 0$

But $\mu(E) = \int_E f\,d\mu + \int_E f\,d\nu$

$\Rightarrow$

$f = 0$

$$0 = \int_E f\,d\nu = \int_{E \cap X_0} \cancel{f}\,d\nu + \int_{E \cap X_+} f\,d\nu$$

But $f > 0$ on $E \cap X_+$

$\Rightarrow \nu(E \cap X_f) = \nu_f(E) = 0$ (otherwise $\int_{E \cap X_f} f \, d\nu > 0$)

$\Rightarrow \nu_f << \mu$

★ Good Exam problem

Ex: $\mu = [0,1]$ $\nu(E) = \delta_{0.5}(E) + \int_E |\sin x| \, dx$